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A nonlinear critical layer generated by the interaction of free Rossby waves

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Two free waves propagating in a parallel shear flow generate a critical layer when their nonlinear interaction induces a perturbation whose phase velocity matches the basic-state velocity somewhere in the flow domain. The condition necessary for this to occur may be interpreted as a resonance condition for a triad formed by the two waves and a (singular) mode of the continuous spectrum associated with the shear. The formation of the critical layer is investigated in the case of freely propagating Rossby waves in a two-dimensional inviscid flow in a β-channel.

A weakly nonlinear analysis based on a normal-mode expansion in terms of Rossby waves and modes of the continuous spectrum is developed; it leads to a system of amplitude equations describing the evolution of the two Rossby waves and of the modes of the continuous spectrum excited during the interaction. The assumption of weak nonlinearity is not however self-consistent: it breaks down because nonlinearity always becomes strong within the critical layer, however small the initial amplitudes of the Rossby waves. This demonstrates the relevance of nonlinear critical layers to monotonic, stable, unforced shear flows which sustain wave propagation.

A nonlinear critical-layer theory is developed that is analogous to the well-known theory for forced critical layers. Differences arise because of the presence of the Rossby waves: the vorticity in the critical layer is advected in the cross-stream direction by the oscillatory velocity field due to the Rossby waves. An equation is derived which governs the modification of the Rossby waves that results from their interaction; it indicates that the two Rossby waves are undisturbed at leading order. An analogue of the Stewartson–Warn–Warn analytical solution is also considered.

1. Introduction

Critical-layer theory is an important element in the study of waves and instabilities in parallel shear flows (see e.g. the reviews by Stewartson 1981 and Maslowe 1986). Of particular interest is the inviscid dynamics of Rossby-wave critical layers, which has attracted considerable attention since the mid-seventies because of its geophysical relevance. Indeed, the strong inhomogeneity of critical-layer flows, with the coexistence of linear and nonlinear regions, has much in common with Rossby-wave breaking events observed in the atmosphere (e.g. Haynes 1989 and references therein).

In the context of two-dimensional flows on the β-plane, critical-layer behaviour is generally manifested in two distinct situations: in unstable flows, when the marginally stable mode possesses a critical level; and in forced flows, when the phase velocity of the forcing locally matches the flow velocity. A comprehensive analysis of these

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two situations was provided by Brown & Stewartson (1978), and by Stewartson (1978) and Warn & Warn (1978), respectively. Using matched asymptotics, they developed simplified equations describing the nonlinear evolution of the critical layer. Subsequent work focused on the forced critical layer, analysing a particular analytical solution found by Stewartson (1978) and Warn & Warn (1978) (referred to as the SWW solution; e.g. Killworth & McIntyre 1985) or using numerical simulations to investigate more general parameter settings (e.g. Haynes 1989).

In this paper, we shall be concerned with a third situation in which nonlinear critical layers are relevant: unforced, stable, shear flows. Tung (1983) considered the evolution of disturbances in a linear shear flow on an infinite β-plane. He concluded (and his conclusion can be extended to any monotonic shear, as pointed out by Brunet & Haynes 1995) that the disturbance dynamics is essentially linear if it is initially so. This is because the disturbance is a sheared disturbance (e.g. Haynes 1987) for which the growth of the vorticity gradient is compensated by a decrease of the streamfunction and the alignment of streamlines with constant-vorticity lines. Tung's analysis relied on certain hypotheses. When these are relaxed it is possible for nonlinear effects to become important in stable shear flows. A first (implicit) hypothesis concerns the nature of the initial condition: Haynes (1987) showed that nonlinearity becomes important if disturbances with very short cross-stream wavelengths are excited, and, as a consequence, that sheared disturbances are unstable. A second hypothesis is that of a monotonic shear flow. For a parabolic jet with weak absolute-vorticity gradient, Brunet & Warn (1990) showed that the dynamics of disturbances always becomes nonlinear in a narrow region (a critical layer) at the jet maximum, regardless of the initial amplitude of the disturbance. Brunet & Haynes (1995) derived a simplified equation describing the dynamics within this critical layer, integrated it numerically, and found that coherent structures are formed at the tip of the jet. A third hypothesis for Tung's (1983) conclusion about nonlinear effects in shear flows is that of an unbounded domain in the cross-stream direction, which implies a basic-flow velocity going from $-\infty$ to $+\infty$. As a result, there are no freely propagating waves or, in other words, no regular normal modes. When the basic-flow velocity is bounded and when there is a β-effect, regular normal modes can exist in the form of Rossby waves. We show in this paper that this leads to significant nonlinear effects in flows which are only weakly disturbed.

We consider a monotonic shear flow $U(y)$ in a channel with an initial disturbance that consists of two free Rossby waves, with wavenumbers k_1 and k_2 and frequencies ω_1 and ω_2. These Rossby waves are such that the term produced by their nonlinear interaction, with frequency $\omega_1 + \omega_2$ and wavenumber $k_1 + k_2$, has a phase velocity matching the flow velocity at some location $y = y_\star$, which may be regarded as a critical level. Assuming weak amplitudes for the Rossby waves, the evolution of the disturbance can be analysed using perturbative approaches. A straightforward regular perturbation expansion indicates the secular growth of the second-order vorticity in the vicinity of $y_\star$ and thus breaks down for long time. More sophisticated perturbative approaches are therefore necessary. Two such approaches are developed in the paper: a weakly nonlinear analysis, which extends the techniques used to study wave-triad interactions (e.g. Craik 1985), and a critical-layer analysis, which employs matched-asymptotics techniques.

The weakly nonlinear analysis is motivated by the analogy between the Rossby-wave interaction considered here and standard wave-triad interactions. As is well-known, a normal-mode approach in a shear flow indicates that, in addition to a discrete spectrum of regular modes (the Rossby waves), there is for each streamwise wavenumber k a continuous spectrum of (singular) modes, with phase velocities in the range of the basic-flow velocity – a superposition of such singular modes represents a sheared disturbance. Ignoring the difficulties associated with the singularities of these modes and the continuous nature of the spectrum, one may interpret the Rossby-wave interaction under study as the resonant interaction between two Rossby waves and a singular mode, namely that with phase velocity $U(y_\star)$. In the light of this interpretation, it seems interesting to attack the problem using an approach that parallels as much as possible the approach employed for resonant wave triads. A first step in this direction was taken in an earlier paper (Vanneste 1996), which describes a technique for studying weakly nonlinear interactions in shear flows including both the Rossby waves and the continuous spectrum. This technique uses recent results about the continuous spectrum due to Balmforth & Morrison (1998). It yields evolution equations for the amplitudes of the Rossby waves as well as for the amplitudes of the singular modes forced by the interactions between the two Rossby waves. These equations were used to examine the Rossby-wave interaction problem, but only at a quasi-linear level, i.e. when the feedback of the forced singular modes onto the waves can be neglected. It was concluded that a singularity forms in the long-time limit at the critical level $y = y_\star$. However, by analogy with wave-triad interactions, one might expect the formation of a singularity to be suppressed if the feedback is retained. The weakly nonlinear analysis developed here investigates this possibility by extending the previous work to include the effect of the feedback. It is shown that this effect is in fact too weak to stop the singularity formation. This indicates that the weakly nonlinear theory cannot remain self-consistent (by contrast with the situation for wave-triad interactions), and corresponds physically to the development of a strongly nonlinear critical layer in the vicinity of $y_\star$. Specifically, if the initial amplitudes of the Rossby waves are proportional to the small parameter ϵ, the flow becomes fully nonlinear after a time proportional to ϵ^{-1} in a critical layer of width proportional to ϵ.

To study the nonlinear evolution of this critical layer in detail, we use matched asymptotics and develop a critical-layer theory analogous to that of Stewartson (1978) and Warn & Warn (1978). To a first approximation, one can interpret the critical layer as resulting from an internal forcing – associated with the nonlinear interaction between the Rossby waves – instead of the standard boundary forcing. However, the presence of Rossby waves in the flow has an important consequence: the vorticity in the critical layer is advected in the cross-stream direction by the Rossby-wave-induced velocity field. Moreover, because we consider a free initial-value problem, the Rossby waves are disturbed by their interaction and the presence of a critical layer. This disturbance is however small; a detailed calculation shows that the Rossby waves amplitudes are unchanged to leading order on the time scale relevant for the critical-layer dynamics. An analogue of the SWW solution is discussed in order to illustrate the differences between the critical layer generated by Rossby-wave interaction and the forced critical layer. It is emphasized that this solution cannot be obtained rigorously, as the long-wave limit on which it rests drastically changes the nature of the Rossby-wave propagation, invalidating our analysis.

The plan of the paper is as follows. In § 2 the Rossby-wave interaction model is described, and the conditions necessary for the formation of a critical layer are discussed. In § 3 a straightforward regular perturbation expansion is performed and it is demonstrated that it breaks down regardless of the weakness of the initial Rossby-wave excitation. Weakly nonlinear amplitude equations are derived in § 4. A truncated system of amplitude equations is then used to study the Rossby-wave interaction and it is shown that the evolution does not remain weakly nonlinear, for a critical layer develops. In § 5 simplified equations governing the critical-layer dynamics and the perturbation of the Rossby waves are derived using matched asymptotics, and the analogue of the SWW solution is considered in § 6. The paper concludes with a discussion in § 7.

2. Formulation

2.1. *Governing equations*

We begin with the vorticity equation for two-dimensional flows in a β-channel and consider the evolution of a disturbance to a steady parallel flow $U(y)$. Scaling the streamwise (zonal) coordinate x by a characteristic length scale L, the cross-stream (meridional) coordinate y by the width of the channel D, the velocity by the range of basic-flow velocity ΔU, and time by $L/\Delta U$, the equation governing the evolution of the disturbance may be written

$$(\partial_t + U\partial_x)\, q + Q'\partial_x\psi + \epsilon\partial(\psi, q) = 0, \tag{2.1}$$

where the disturbance vorticity q and the disturbance streamfunction ψ are related by

$$q = \nabla^2\psi := (\mu^2\partial_{xx}^2 + \partial_{yy}^2)\psi,$$

with $\mu := D/L$. The associated boundary conditions of no-normal flow and conservation of circulation (e.g. Pedlosky 1987, § 3.25) read

$$\partial_x\psi = 0 \quad \text{and} \quad \partial_t \int \partial_y\psi \, \mathrm{d}x = 0 \quad \text{at } y = 0, 1.$$

In (2.1), $Q' := \beta - U''$ (with $' := \mathrm{d}/\mathrm{d}y$) is the basic vorticity gradient and $\epsilon \ll 1$ characterizes the disturbance amplitude. The non-dimensional parameter β, related to its dimensional counterpart $\tilde{\beta}$ through $\beta = D^2\tilde{\beta}/\Delta U$, is assumed to be of order one. We also assume that the basic flow satisfies

$$U' > 0 \quad \text{and} \quad Q' > 0 \quad \text{for } y \in [0, 1].$$

The first condition ensures the monotonicity of the basic flow, and the second condition its nonlinear stability (as proved using Arnold's energy–Casimir method; see e.g. Holm *et al.* 1985). For convenience, one can take advantage of the translational invariance of (2.1) in x and fix the minimum and maximum basic velocities in the channel as $U_\mathrm{m} := U(0) = 0$ and $U_\mathrm{M} := U(1) = 1$.

Introducing modal solutions $q = q_k(y)\exp[\mathrm{i}k(x - ct)]$ in the linearization of (2.1) yields the Rayleigh–Kuo equation which, for this geometry, admits a discrete spectrum of Rossby waves with phase velocities $c_{k,n} < U_\mathrm{m}$, $n = 1, 2, \ldots$, and a continuous spectrum of modes with $U_\mathrm{m} < c < U_\mathrm{M}$ (see Appendix A). The modes of the continuous spectrum are singular at their critical level y_c defined by $U(y_c) = c$; in monotonic basic flows, the critical level position can be used instead of the phase velocity to identify each singular mode.

Consider now two Rossby waves with wavenumbers k_1, k_2 and indices n_1, n_2, and thus with frequencies $\omega_1 = k_1\, c_{k_1,n_1}$ and $\omega_2 = k_2\, c_{k_2,n_2}$. Through nonlinear interaction they excite modes with wavenumber $k_\star$ satisfying

$$k_\star + k_1 + k_2 = 0. \tag{2.2}$$

(By convention, we consider sum interactions only, allowing for both positive and negative values of each wavenumber.) Among these modes, those belonging to the continuous spectrum are significantly excited provided that

$$\exists\, y_\star \in [0, 1] \;:\quad k_\star U(y_\star) + \omega_1 + \omega_2 = 0, \tag{2.3}$$

i.e. provided that a singular mode, with critical level position $y_c = y_\star$, forms a resonant triad with the two Rossby waves. Equations (2.2) and (2.3) define the type of interactions studied in this paper.

2.2. *Interaction conditions*

Restrictions on the wavenumbers participating in the interaction can be derived from (2.2)–(2.3) by noting that

$$\frac{k_1}{c_2 - U(y_\star)} = \frac{k_2}{U(y_\star) - c_1} = \frac{k_\star}{c_1 - c_2}.$$

With $U_{\mathrm{m}} = 0$, one can choose $c_1 < c_2 < 0$, whence $k_1 k_\star > 0$ and $k_2 k_\star < 0$. Without loss of generality, k_1 can be taken positive to give the condition

$$k_1 > 0, \quad k_2 < 0, \quad k_\star > 0, \tag{2.4}$$

which indicates that one of the Rossby waves always has the largest wavenumber (in absolute value). Conditions (2.2)–(2.3) are analogous to the resonant interaction conditions in the standard three-wave interaction, but with a dispersion relation containing two distinct parts: one associated with the Rossby waves, given by $\omega = \omega(k) = kc_{k,n}$, and the other associated with singular modes, given by the double inequality $kU_{\mathrm{m}} < \omega < kU_{\mathrm{M}}$ and thus corresponding to a sector in the (k, ω)-plane. The standard graphical construction used to locate resonant triads (e.g. Simmons 1969) can be adapted for the interaction between two Rossby waves and a singular mode. This is illustrated in figure 1, which displays a typical Rossby-wave dispersion relation in the (k, ω)-coordinate system and the (shaded) sector associated with singular modes in another coordinate system denoted by $(k_{\mathrm{s}}, \omega_{\mathrm{s}})$. This system has its origin at a point (k_1, ω_1) of the Rossby-wave dispersion curve. Any other point $(|k_2|, \omega_2)$ (represented by a dot) on the dispersion curve lying in the shaded area forms a resonant triad with the first Rossby wave and a singular mode. Indeed, taking (2.4) into account, it can be verified on the figure that the singular mode with wavenumber $k_{\mathrm{s}} = k_\star$ and phase velocity $U(y_\star)$ given by the slope of the dashed line satisfies (2.2)–(2.3). Because the singular modes belong to a continuous rather than discrete spectrum, a single given Rossby wave forms an infinite number of resonant triads involving singular modes. This is in contrast with the standard three-wave interaction and can be particularly important for domains periodic in the x-direction, when most Rossby waves cannot be involved in regular wave triads because of the wavenumber discretization.

3. Regular perturbation expansion

We now study the evolution of a disturbance that initially consists of two weak-amplitude Rossby waves satisfying (2.2)–(2.3). A regular perturbation expansion is first used to examine this evolution. Although such an expansion can be expected to break down at some point, it is useful to consider it in detail: the solution it yields describes the early evolution of the system, and the nature of the breakdown guides the derivation of the singular perturbation theory which is developed in § 5. We thus look for a solution of (2.1) given by the expansion

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \cdots \quad \text{and} \quad q = q^{(0)} + \epsilon q^{(1)} + \cdots, \tag{3.1}$$

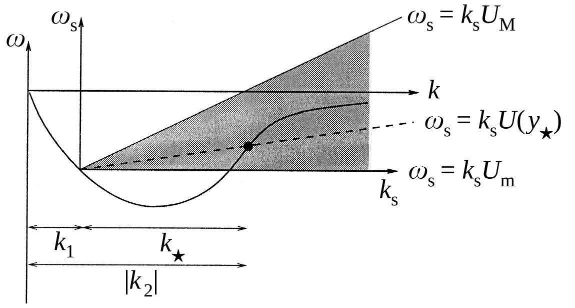

FIGURE 1. Graphical solution of the equations (2.2)–(2.3) for the resonance of two Rossby waves and a singular mode. A typical dispersion relation for Rossby waves is displayed in the (k, ω)-coordinate system (solid curve), while the singular modes are located in the (k_s, ω_s)-coordinate system (shaded sector; see text for details).

where the leading-order term is the superposition of two Rossby waves:

$$\psi^{(0)} = \mathrm{Re}\left[R_1\,\psi_1(y)\,\mathrm{e}^{\mathrm{i}\theta_1} + R_2\,\psi_2(y)\,\mathrm{e}^{\mathrm{i}\theta_2}\right], \tag{3.2}$$

with $\theta_j := k_j x - \omega_j t$ and $\psi_j := \psi_{k_j,n_j}$, $j = 1, 2$. The amplitudes R_1, R_2 are fixed by the initial condition. It is convenient to use a reference frame moving at velocity $U(y_\star)$, so that (2.3) reduces to

$$\omega_1 + \omega_2 = 0, \quad \text{with} \quad U(y_\star) = 0.$$

Introducing (3.2) into (2.1) leads at $O(\epsilon)$ to the inhomogeneous equation

$$(\partial_t + U\partial_x)\,q^{(1)} + Q'\partial_x\psi^{(1)} = \tfrac{1}{2}\mathrm{Re}\big[(R_1R_2)^* f_+(y)\,\mathrm{e}^{\mathrm{i}k_\star x} + R_1(R_2)^* f_-(y)\,\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)} + R_1^2\,g_1(y)\,\mathrm{e}^{2\mathrm{i}\theta_1} + R_2^2\,g_2(y)\,\mathrm{e}^{2\mathrm{i}\theta_2}\big]. \tag{3.3}$$

The functions f_+, f_-, g_1 and g_2 are defined in terms of ψ_1 and ψ_2; in particular,

$$f_+(y) := \mathrm{i}\left[k_1\psi_1(q_2)' - k_2(\psi_1)'q_2 + k_2\psi_2(q_1)' - k_1(\psi_2)'q_1\right],$$

where $q_j := \psi_j'' - k_j^2\psi_j$. The phase velocities corresponding to the last three terms of (3.3) are all smaller than the minimum velocity in the channel (see (2.4)); the response of $\psi^{(1)}$ to these terms can therefore be computed without difficulty if we assume they are not resonant with free Rossby waves. The first term on the right-hand side of (3.3), by contrast, has a zero phase velocity and is thus associated with a critical level at $y = y_\star$. We now focus on the response to this term.

In Vanneste (1996), this response was computed using an expansion in terms of the singular modes with wavenumber $k_\star$. A time-dependent expression was derived for the response streamfunction, which was shown to tend to a stationary structure of the form $\mathrm{Re}[\phi(y)\exp(\mathrm{i}k_\star x)]$ in the long-time limit (see equation (4.9) in that paper). Here, we derive this stationary structure directly, in a form that will be more suited to the analysis of § 5. From (3.3), ϕ satisfies the equation

$$U(y)\left(\phi'' - \mu^2k_\star^2\phi\right) + Q'\phi = h, \quad \text{with} \quad \phi(0) = \phi(1) = 0, \tag{3.4}$$

where $h := -\mathrm{i}(R_1R_2)^* f_+/(2k_\star)$. The method of variation of parameters can be employed to obtain ϕ. Using two independent Frobenius solutions of the homogeneous version of (3.4) on each side of the critical level, ϕ is expressed in terms of four constants. The boundary conditions at $y = 0, 1$ as well as continuity at the critical level provide three relations between these constants. As usual, a fourth relation is found by considering the velocity jump

$$\left[\frac{\mathrm{d}\phi}{\mathrm{d}y}\right]_{y_\star^-}^{y_\star^+} := \lim_{\epsilon\to 0^+}\left(\left.\frac{\mathrm{d}\phi}{\mathrm{d}y}\right|_{y_\star+\epsilon} - \left.\frac{\mathrm{d}\phi}{\mathrm{d}y}\right|_{y_\star-\epsilon}\right).$$

The velocity jump is derived from the linearized evolution equation for the vorticity which remains transient for all time (cf. Stewartson 1978). In the long-time limit this equation shows that the vorticity grows linearly in time in the vicinity of the critical level; the velocity jump, however, is bounded and given by

$$\left[\frac{\mathrm{d}\phi}{\mathrm{d}y}\right]_{y_\star^-}^{y_\star^+} = \mathrm{i}\pi\frac{h_\star - Q'_\star\phi(y_\star)}{U'_\star}, \tag{3.5}$$

where the subscripts $\star$ denote functions evaluated for $y = y_\star$. Identifying (3.5) with the expression for the velocity jump derived from the Frobenius solutions provides the fourth relation between the constants which are then completely determined.

It proves convenient to decompose ϕ into two parts so as to isolate the contribution due to the internal forcing h and that due to velocity jump imposed at $y = y_\star$ by the vorticity dynamics. The first part, ϕ^{f} say, is thus defined as the response to the forcing h with an imposed zero velocity jump at the critical level; the other part corresponds to a free solution with an imposed velocity jump. This latter part can be recognized as a multiple of the singular mode $\psi_{k_\star}(y;y_\star)$ defined in Appendix A, so that the complete solution takes the compact form

$$\phi(y) = \phi^{\mathrm{f}}(y) + C\,\psi_{k_\star}(y;y_\star), \tag{3.6}$$

where the arbitrary constant C is determined by the velocity jump (3.5). By definition of ϕ^{f} and of the singular modes, (3.6) yields

$$\left[\frac{\mathrm{d}\phi}{\mathrm{d}y}\right]_{y_\star^-}^{y_\star^+} = \lambda_{k_\star}(y_\star)C, \tag{3.7}$$

where, as detailed in Appendix A, $\lambda_{k_\star}(y_\star) = \left[\mathrm{d}\psi_{k_\star}(y;y_\star)/\mathrm{d}y\right]_{y_\star^-}^{y_\star^+}$ is fixed by the normalization chosen for $\psi_{k_\star}(y;y_\star)$. Equations (3.5)–(3.7) lead to an expression for C.

The solution just developed is clearly not uniformly valid in time: the vorticity increases linearly in time near the critical level $y = y_\star$, and thus neglected nonlinear terms can be expected to become important for $t = O(\epsilon^{-1})$.† An important factor neglected by the regular expansion is the feedback of the response ϕ (i.e. the excited singular modes) on the Rossby waves. The role of this feedback is analysed in the next section using a weakly nonlinear approach.

4. Weakly nonlinear analysis

As already mentioned, the problem under consideration can be interpreted as the resonant interaction of two Rossby waves with a singular mode of the continuous spectrum. This interaction presents similarities with standard wave-triad interactions in the light of which it does not come as a surprise that the regular perturbation expansion of the previous section fails for $t = O(\epsilon^{-1})$: on such time scale, one might

† In the particular case where $h_\star = Q'_\star\phi^{\mathrm{f}}_\star$, the vorticity does not increase, and therefore the linear solution does not break down in the long-time limit. We do not consider this situation in what follows.

expect the system to be governed by equations similar to the three-wave equations governing wave triads, i.e. weakly nonlinear equations that take into account the full coupling between the Rossby waves and the singular modes. In this section, we derive such equations and analyse their behaviour. This analysis proves instructive although it turns out that the weakly nonlinear equations are not self-consistent: they predict that the nonlinearity does not remain weak because of the formation of a critical layer.

A technique for deriving weakly nonlinear equations for disturbances in shear flows was described in Vanneste (1996). It relies on the normal-mode expansion reviewed in Appendix A and leads to evolution equations for the amplitudes of the Rossby waves and the singular modes. In principle, the technique is straightforward: the expansion (A 4) with time-dependent amplitudes $A_{k,n}(t)$ and $A_k(y_c;t)$ is introduced into the nonlinear evolution equation (2.1), and the orthogonality relations (A 5) are used to project the resulting equation onto each mode. As a result, equations of the following form are obtained for the Rossby-wave and singular mode amplitudes:

$$\partial_t A_{k,n}(t) = \epsilon \text{ n.l.t.} \quad \text{and} \quad \partial_t A_k(y_c;t) = \epsilon \text{ n.l.t.}, \tag{4.1}$$

where n.l.t. denotes nonlinear terms.

In order to use these equations practically, one must truncate the infinite-dimensional system they constitute and derive simplified systems which retain the essence of a particular interaction. In particular, the simplified system we are interested in should consist of equations for the two excited Rossby waves and for the singular modes with wavenumber $k_\star$; all these singular modes must be taken into account since truncation to a single one (e.g. the one with critical level at $y_\star$) would lead to singular integrals. It should however be realized that the truncation cannot be made directly: indeed, due to the presence of $\partial_y q$ in the nonlinearity of (2.1) and of the Dirac distribution in the singular mode vorticity, terms of the form

$$\begin{aligned} A_{k',n}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{k',n}t} \int_0^1 \delta'(y-y_c) A_k(y_c,t)\,\mathrm{e}^{-\mathrm{i}kU(y_c)t} F(k,k',n,y_c)\,\mathrm{d}y_c\ \mathrm{e}^{\mathrm{i}(k+k')x} \\ = -\mathrm{i}kU'(y)\,t\,A_{k',n}(t) A_k(y,t) F(k,k',n,y)\,\mathrm{e}^{-\mathrm{i}(kU(y)+\omega_{k',n})t}\,\mathrm{e}^{\mathrm{i}(k+k')x} + \cdots \end{aligned}$$

appear when the normal mode expansion is introduced into (2.1). Here, F is a smooth function whose precise form is unimportant. After projection, these terms lead to explicit linear time dependences in the amplitude equations for the singular modes, and thus to the presence of secularities even in the absence of resonance. Physically, these terms correspond to the cross-stream advection of the vorticity associated with singular modes by the Rossby-wave velocity field; they are secular because the vorticity gradient of a superposition of singular modes is growing linearly in time.†

To eliminate these secularities, we introduce a variable transformation from q to $\tilde{q}$, where

$$\tilde{q} := q - \epsilon \partial_y \left(\frac{q^2}{2Q'} \right). \tag{4.2}$$

As detailed below, this transformation leads to an equation for the vorticity-like quantity $\tilde{q}$ with the same linear part as the equation for q, but with a modified nonlinear part that does not contain y-derivatives of $\tilde{q}$. Thanks to this modification,

† More complex time dependences arise from the self-advection of singular modes; but since the superposition of these modes represents sheared disturbances, Tung's (1983) argument can be employed to show that the corresponding contributions decrease when $t = O(\epsilon^{-1})$ for most initial conditions. Therefore they do not affect the truncation process.

the secularities are removed from (4.1). For the transformation to be one-to-one, it is necessary that

$$\epsilon|\partial_y q| \ll Q', \tag{4.3}$$

which is a statement of weak nonlinearity and a condition for positive cross-stream gradient of absolute vorticity. This condition is likely to be violated for long time, in which case it is not obvious whether any weakly nonlinear theory can remain relevant. It is nevertheless possible to obtain a version of (4.2) which would be valid for longer time. We defer the description of this transformation to the Discussion, § 7; the main purpose of this section is to obtain a generic form of the amplitude equations governing the interaction between two Rossby waves and the continuous spectrum, and this form does not depend on the precise expression of the variable transformation. Taking the time derivative of (4.2), using (2.1) and condition (4.3), and neglecting terms of $O(\epsilon^2)$ and higher, one derives an evolution equation for $\tilde{q}$ of the form

$$(\partial_t + U\partial_x)\,\tilde{q} + Q'\partial_x\psi - \epsilon\,\partial_x\left(\tilde{q}\partial_y\psi + \frac{U'}{2Q'}\tilde{q}^2\right) = 0, \tag{4.4}$$

where the streamfunction is derived from the approximate relation

$$\nabla^2\psi = \tilde{q} + \epsilon\,\partial_y\left(\frac{\tilde{q}^2}{2Q'}\right).$$

The evolution equation for $\tilde{q}$ does not contain y-derivatives of $\tilde{q}$. Thus, the secularities associated with the growing vorticity gradient of superpositions of singular modes have been removed. Equation (4.4) is also such that the $k = 0$ component of $\tilde{q}$ is invariant, i.e. there is no wave–mean flow interaction for (4.4). We can now expand $\tilde{q}$ rather than q in normal modes according to (A 4) and apply the procedure leading to amplitude equations. These are again given by (4.1) (because the linearized equation is the same for q and $\tilde{q}$), but without linearly growing terms on the right-hand side. A truncation is then possible, provided that the singular mode amplitudes $A_k(y_c\,;t)$ are smooth functions of y_c. We shall see that this assumption does not remain valid in our problem, indicating the formation of a critical layer where the nonlinearity is not weak.

The truncated system governing the interaction between the two Rossby waves (with amplitudes $A_1(t) := A_{k_1,n_1}(t)$ and $A_2(t) := A_{k_2,n_2}(t)$) and the singular modes (with amplitudes $A_{k_\star}(y_c\,;t)$) has the form

$$\left.\begin{aligned}
\partial_t A_1(t) &= \epsilon[A_2(t)]^* \int_0^1 I_1(y_c)[A_{k_\star}(y_c\,;t)]^*\,\mathrm{e}^{\mathrm{i}\Omega(y_c)t}\,\mathrm{d}y_c,\\
\partial_t A_2(t) &= \epsilon[A_1(t)]^* \int_0^1 I_2(y_c)[A_{k_\star}(y_c\,;t)]^*\,\mathrm{e}^{\mathrm{i}\Omega(y_c)t}\,\mathrm{d}y_c,\\
\partial_t A_{k_\star}(y_c\,;t) &= \epsilon I_{k_\star}(y_c)[A_1(t)A_2(t)]^*\,\mathrm{e}^{\mathrm{i}\Omega(y_c)t},
\end{aligned}\right\} \tag{4.5}$$

which is a direct extension of the standard three-wave equations. The corresponding initial conditions are simply $A_1(0) = R_1$, $A_2(0) = R_2$ and $A_{k_\star}(y_c\,;t) = 0, \forall y_c \in [0,1]$. In the above equation, $\Omega(y_c) := k_\star U(y_c) + \omega_1 + \omega_2$ satisfies $\Omega(y_\star) = 0$. The interaction coefficients I_1, I_2 and $I_{k_\star}$ are functions of the critical level position y_c; they are defined by y-integrals of the structures $\psi_1(y)$, $\psi_2(y)$ and $\psi_{k_\star}(y\,;y_c)$, but their explicit expressions are not necessary here, since we examine only the qualitative behaviour of (4.5). It is important to note that (pseudo)energy and (pseudo)momentum conservation and the

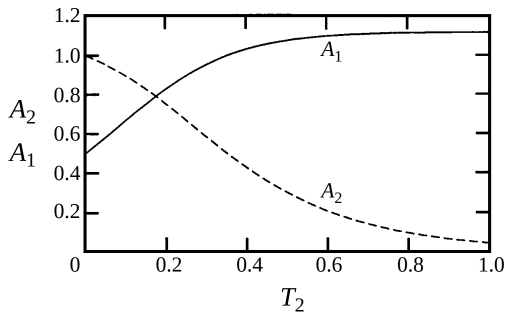

FIGURE 2. Time evolution of the amplitudes $A_1(T_2)$ and $A_2(T_2)$ of the interacting Rossby waves as predicted by the weakly nonlinear theory.

fact that $Q' > 0$ (or equivalently nonlinear stability) require $k_1 I_1(y_\star), k_2 I_2(y_\star), k_\star I_{k_\star}(y_\star)$ to have the same sign (cf. Ripa 1981; Vanneste & Vial 1994). Together with (2.4), this indicates that $I_2(y_\star)$ is oppositely signed to the other two interaction coefficients.

Using (4.5), we can now assess whether the feedback of the singular modes on the Rossby waves is sufficient to limit the growth of the singular modes – that is, whether the interaction leads to balanced weakly nonlinear dynamics, as is the case for standard wave triads. Simple scaling arguments allow us to conclude that this is not so. Let $T_\alpha := \epsilon^\alpha t$, where α is a constant to be determined, be the slow time relevant to the weakly nonlinear dynamics. It is clear that, due to phase mixing, only the singular modes with $Y_\alpha := (y_c - y_\star)/\epsilon^\alpha = O(1)$ contribute to the integrals in (4.5). Rewriting (4.5) in terms of T_α and Y_α, we can expand the interaction coefficients and extend the range of integration to obtain the approximate system

$$\left.\begin{aligned} \partial_{T_\alpha} A_1(T_\alpha) &= \epsilon [A_2(T_\alpha)]^* I_1(y_\star) \int_{-\infty}^{\infty} [A_{k_\star}(Y_\alpha; T_\alpha)]^* \, \mathrm{e}^{\mathrm{i}\Omega'_\star Y_\alpha T_\alpha} \, \mathrm{d}Y_\alpha, \\ \partial_{T_\alpha} A_2(T_\alpha) &= \epsilon [A_1(T_\alpha)]^* I_2(y_\star) \int_{-\infty}^{\infty} [A_{k_\star}(Y_\alpha; T_\alpha)]^* \, \mathrm{e}^{\mathrm{i}\Omega'_\star Y_\alpha T_\alpha} \, \mathrm{d}Y_\alpha, \\ \partial_{T_\alpha} A_{k_\star}(Y_\alpha; t) &= \epsilon^{1-\alpha} I_{k_\star}(y_\star) [A_1(T_\alpha) A_2(T_\alpha)]^* \, \mathrm{e}^{\mathrm{i}\Omega'_\star Y_\alpha T_\alpha}. \end{aligned}\right\} \quad (4.6)$$

Regardless of the value of α, the first two equations indicate that a nonlinear balance is possible only if $A_{k_\star}(Y_\alpha; T_\alpha) = O(\epsilon^{-1})$, i.e. if the singular modes have a large amplitude. The last equation of (4.6) then shows that the proper scaling is given by $\alpha = 2$, indicating that nonlinear balance is achieved for $t = O(\epsilon^{-2})$. At that time, the weakly nonlinear equations are not valid, since the singular mode amplitudes $A_{k_\star}(Y_\alpha; T_\alpha)$ are large. This indicates that the weakly nonlinear theory does not remain self-consistent.

This scaling argument can be verified by a direct integration of (4.5) or (4.6). This integration can be performed numerically or, in the case of (4.6), analytically. A typical solution, obtained with initial conditions $A_1(0) = R_1 = 0.5, A_2(0) = R_2 = 1$, with $I_1(y_\star) = -I_2(y_\star) = I_{k_\star} = 1$ and $\Omega'_\star = 1$ is displayed in figures 2 and 3. The scaled coordinates T_2 and Y_2 and amplitudes $\hat{A}_{k_\star} = \epsilon A_{k_\star}$ are employed. For $t = O(\epsilon^{-1})$ the Rossby wave amplitudes are almost unaffected by their interaction with the singular modes. As a consequence, the amplitude of the singular modes with $y_c \approx y_\star$ continues to grow. It is only when the singular mode amplitude is very large ($A_{k_\star}(y_\star; t) = O(\epsilon^{-1})$) and thus when the weakly nonlinear analysis has broken down that, according to (4.6), the growth stops and a steady state is attained.

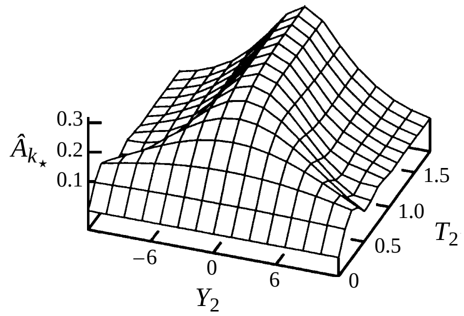

FIGURE 3. Time evolution of the amplitude $\hat{A}_{k_\star}(Y_2; T_2) = \epsilon A_{k_\star}(Y_2; T_2)$ of the singular modes as predicted by the weakly nonlinear theory. Only a narrow band of modes is excited as appears from the relation $y_c = y_\star + \epsilon^2 Y_2$ between the critical level position y_c and the scaled coordinate Y_2. Note that saturation occurs for $A_{k_\star}(0, T_2) = O(\epsilon^{-1}) \gg 1$.

The breakdown of the weakly nonlinear theory indicates that the free evolution of the flow leads to strong nonlinearity. Since this nonlinearity is confined within a narrow critical layer surrounding $y = y_\star$, we can use matched asymptotics to derive a simplified equation governing the long-time evolution of the flow, as done by Stewartson (1978) and Warn & Warn (1978) in their study of the forced Rossby-wave critical layer.

5. Critical-layer analysis

A preliminary step for the critical-layer analysis is the determination of the temporal and spatial scales relevant for the nonlinear evolution of the critical layer. These scales can be obtained by examining the nature of the breakdown of the regular perturbation expansion or of the weakly nonlinear analysis. Let us first return to the regular perturbation expansion of § 3. The linearized vorticity equation indicates that the vorticity grows linearly with time in a critical layer whose width decreases like t^{-1}. Proceeding with the regular expansion, we compute the forcing term $\psi_x^{(0)} q_y^{(1)}$ in the equation for $q^{(2)}$ and obtain a time dependence of the type $t^2 \exp(-\mathrm{i}\omega_n t)$ within the critical layer. Therefore, the dominant behaviour of $q^{(2)}$ is found to have the form $t^2 \exp(-\mathrm{i}\omega_n t)$. We thus conclude that the regular expansion (3.1) breaks down for $t = O(\epsilon^{-1})$ in a layer of width ϵ.† In fact, the same conclusion can be drawn from the weakly nonlinear analysis of § 4, since it is essentially equivalent to the linear one for $t < O(\epsilon^{-1})$. The weakly nonlinear theory breaks down because it assumes a smooth dependence of $A_{k_\star}$ on y_c, an assumption which ceases to hold for $t = O(\epsilon^{-1})$.

We start the critical-layer analysis by defining the slow time

$$T := \epsilon t$$

which changes by $O(1)$ during the nonlinear evolution. It is again convenient to use a reference frame such that $U(y_\star) = 0$ and therefore $\omega_1 = -\omega_2$. Because various frequencies are present in the system (ω_1, ω_2 and their harmonics), the fast time t cannot be entirely removed as is the case for the forced critical layer. Yet, we can

† The time after which the regular expansion breaks down and the corresponding critical-layer width found here are different from those found in the standard forced critical layer ($O(\epsilon)$ vs. $O(\epsilon^{1/2})$). This is because the forcing associated with the interacting Rossby waves is $O(\epsilon^2)$, whereas the external forcing is $O(\epsilon)$ in the standard critical layer.

neglect the transients by considering dependences on t of the form $\exp(\mathrm{i}\Omega t)$ only, where $\Omega = m\omega_1 + n\omega_2$ and m, n are integers. In what follows, we use the superscripts s and r to denote the parts of the solution with frequencies 0 and $\omega_1 = -\omega_2$, respectively. The other harmonics are gathered in terms denoted by h. We refer to terms depending only on T as slow, and to those depending also on t as fast.

5.1. *Outer solution*

In the outer region (i.e. $y - y_\star \gg \epsilon$) the regular expansion (3.1) is valid, and at leading order one recovers the superposition of Rossby waves (3.2), with time-dependent amplitudes $R_1(T), R_2(T)$. At the next order, (3.3) is found with the extra term $-\partial_T R_1 q_1 \exp(\mathrm{i}\theta_1) - \partial_T R_2 q_2 \exp(\mathrm{i}\theta_2)$ on the right-hand side. Solvability then requires

$$\partial_T R_1 = \partial_T R_2 = 0.$$

To leading order, the Rossby waves are thus undisturbed by their interaction. An expression for $\psi^{(1)}$ can be written in the general form

$$\psi^{(1)} = \psi^{(1,\mathrm{s})} + \psi^{(1,\mathrm{r})} + \psi^{(1,\mathrm{h})},$$

where

$$\psi^{(1,\mathrm{s})} = \mathrm{Re}\left[\phi^{\mathrm{f}}(y)\,\mathrm{e}^{\mathrm{i}k_\star x} + \sum_{k=1}^{\infty} C_k(T)\psi_k(y; y_\star)\,\mathrm{e}^{\mathrm{i}kx}\right] \tag{5.1}$$

contains all harmonics with the exception of a mean ($k = 0$) component which can be shown to vanish. (Note that we have adopted the notation $\sum_{k=1}^{\infty}$ as a shorthand: assuming that the channel is periodic in x, the sum is in fact taken over all wavenumbers that are multiples of $2\pi/L$, where L is the channel period; if the channel is infinite, the sum must be interpreted as an integral.) The evolution of the coefficients $C_k(T)$ is determined from the dynamics inside the critical layer (as is the case for the forced critical layer). A free solution

$$\psi^{(1,\mathrm{r})} = \mathrm{Re}\left[S_1(T)\psi_1(y)\,\mathrm{e}^{\mathrm{i}\theta_1} + S_2(T)\psi_2(y)\,\mathrm{e}^{\mathrm{i}\theta_2}\right], \tag{5.2}$$

representing a small correction to the initial Rossby waves, is added to $\psi^{(1)}$ so as to cancel secular terms appearing at the next order. An explicit expression for the harmonic term $\psi^{(1,\mathrm{h})}$ can be obtained but we do not describe it here. For our purpose it suffices to note that it is independent of T and smooth at the critical level.

Proceeding with the expansion, one finds at $O(\epsilon^2)$ the equation

$$(\partial_t + U\partial_x)\, q^{(2)} + Q'\partial_x\psi^{(2)} = -\partial_T q^{(1)} - \partial(\psi^{(0)}, q^{(1)}) - \partial(\psi^{(1)}, q^{(0)}) \tag{5.3}$$

whose solution can again be written

$$\psi^{(2)} = \psi^{(2,\mathrm{s})} + \psi^{(2,\mathrm{r})} + \psi^{(2,\mathrm{h})}.$$

The component $\psi^{(2,\mathrm{s})}$ can be computed without difficulty. Near the critical level, $\psi^{(2,\mathrm{s})} \sim \ln|y - y_\star|$, which leads to logarithmic terms in the inner expansion. Although a complete expression for $\psi^{(2)}$ is not strictly necessary as it represents a small correction to $\psi^{(1)}$, it is crucial to consider the derivation of $\psi^{(2,\mathrm{r})}$ carefully, for the solvability condition which will be required determines the evolution of S_1 and S_2 in (5.2). As detailed in Appendix B, the derivation of this solvability condition is not straightforward for two reasons. First, some of the resonant terms are singular at $y = y_\star$, and thus we need to derive the solvability condition for a singular equation; to this end, we employ a method introduced by Benney & Maslowe (1975). This method requires an explicit derivation of $\psi^{(2,\mathrm{r})}$, and this leads to the second difficulty, namely the fact that $\psi^{(2,\mathrm{r})}$ has a discontinuous y-derivative at $y_\star$. Indeed, as will be explicitly demonstrated below, the dynamics inside the critical layer imposes a velocity jump across the critical layer on the fast solution at this order. In Appendix B, we describe a derivation of the solvability conditions taking these elements into account. It leads to two equations of the form

$$\partial_T S_n + P_n(T) = 0, \quad n = 1, 2, \tag{5.4}$$

which govern the evolution of the $O(\epsilon)$ modification of the initial Rossby waves due to their interaction. The P_n have the general form

$$P_n(T) = c_n + d_n[C_{k_\star}(T)]^*,$$

where c_n and d_n are complex constants. This leads to two remarks. First, the S_n are coupled to the critical-layer dynamics through $C_{k_\star}$. Because the critical-layer equation governing the evolution of the C_k does not depend on S_n (see § 5.2), this implies that the amplitudes S_n of the Rossby-wave modification are slaved to the $C_{k_\star}$. Next, (5.4) can be integrated formally, leading to

$$-S_n(T) = \int_0^T P_n(T')\,\mathrm{d}T' = c_n T + d_n \int_0^T \left[C_{k_\star}(T')\right]^*\,\mathrm{d}T'. \tag{5.5}$$

This expression contains a secular term which suggests a break-down of our expansion for $T = O(\epsilon^{-1})$, i.e. $t = O(\epsilon^{-2})$. However, this secularity can be removed by allowing the Rossby-wave amplitudes R_1 and R_2 to depend on $T_2 := \epsilon T = \epsilon^2 t$. On that time scale, the terms c_n simply lead a nonlinear frequency shift of the Rossby waves associated with the presence of harmonics.

5.2. *Inner solution*

Since the width of the critical layer has been estimated as $O(\epsilon)$, we define the stretched coordinate

$$Y = \frac{y - y_\star}{\epsilon},$$

which is $O(1)$ in the inner region. In terms of this variable, the outer solution $\psi^{(0)} + \epsilon\psi^{(1)} + \cdots$ takes the form

$$\psi = \psi_\star^{(0)} + \epsilon\left(Y\,\psi_{y\star}^{(0)} + \psi_\star^{(1)}\right) + O(\epsilon^2 \ln \epsilon). \tag{5.6}$$

This expansion provides the boundary condition for the inner expansion as $Y \to \pm\infty$. In principle, higher-order terms can be evaluated; in particular the $O(\epsilon^2 \ln \epsilon)$ term in (5.6) results from logarithmic terms in $\psi^{(2,\mathrm{s})}$ and $\psi^{(2,\mathrm{r})}$. For the $O(\epsilon^2)$ term, one must calculate the dominant behaviour of $\psi^{(3)}$ near the critical level. These calculations are very similar to those of Warn & Warn (1978), so we omit the details and give only the higher-order term that is crucial for the critical-layer equation, namely that associated with the velocity jump in $\psi^{(1,\mathrm{s})}$. It is obtained from (5.1) and (A 3) and is given by

$$\epsilon^2 \tfrac{1}{2}|Y|\,\mathrm{Re}\left[\sum_{k=1}^{\infty} \lambda_k(y_\star) C_k\,\mathrm{e}^{\mathrm{i}kx}\right], \tag{5.7}$$

corresponding to the velocity jump

$$\left[\frac{\partial \psi^{(1,s)}}{\partial y}\right]_{y_\star^-}^{y_\star^+} = \mathrm{Re}\left[\sum_{k=1}^{\infty} \lambda_k(y_\star) C_k \, \mathrm{e}^{\mathrm{i}kx}\right]. \tag{5.8}$$

We now expand the streamfunction inside the critical layer as

$$\psi = \psi_\star^{(0)}(x,t) + \epsilon\left(Y\psi_{y\star}^{(0)} + \psi_\star^{(1)}\right) + \epsilon^2 \ln\epsilon\, \Psi^{(l,2)}(x,t,T) + \epsilon^2 \Psi^{(2)}(x,Y,t,T) + \cdots .$$

In writing this expansion, we have anticipated the fact that the $O(1)$ and $O(\epsilon)$ terms are entirely fixed by the boundary conditions (5.6). The same holds for $\Psi^{(l,2)}$ which we do not detail here. Introducing this expansion in the nonlinear vorticity equation (2.1) leads to a sequence of evolution equations. At $O(\epsilon^0)$, we find

$$\mathscr{L}\Psi_{YY}^{(2)} + \partial_t \psi_{xx\star}^{(0)} + Q_\star' \partial_x \psi_\star^{(0)} = 0, \tag{5.9}$$

where

$$\mathscr{L} := \partial_t + \partial_x \psi_\star^{(0)} \partial_Y .$$

The operator $\mathscr{L}$, which appears at each order, can be simplified by introducing the new independent variables

$$\tau = t, \quad \xi = x, \quad \eta = Y + \frac{q_\star^{(0)}}{Q_\star'}. \tag{5.10}$$

Using the leading order for equation for $\psi^{(0)}$ at $y = y_\star$ leads to

$$\mathscr{L} = \partial_\tau .$$

The general solution of (5.9) is therefore

$$\Psi_{YY}^{(2)} = \psi_{yy\star}^{(0)} + Z(\xi,\eta,T),$$

where Z is an arbitrary function. The leading-order vorticity within the critical layer, given by $\psi_{xx\star}^{(0)} + \Psi_{YY}^{(2)}$, is thus the sum of the Rossby-wave vorticity $q_\star^{(0)}$ and of $Z(\xi,\eta,T)$, the latter being the leading-order vorticity induced by the Rossby-wave interaction. Z is the central quantity for the critical-layer dynamics; interestingly, it is not entirely slow, as dependence on the fast time t is contained in the variable transformation (5.10). An evolution equation for Z is now derived.

At $O(\epsilon)$, we find the equation

$$\begin{aligned}\mathscr{L}\Psi_{YY}^{(3)} &+ Y\left(\partial_t \psi_{xxy\star}^{(0)} + U_\star' \partial_x q_{y\star}^{(0)} + Q_\star' \partial_x \psi_{y\star}^{(0)} + Q_\star'' \partial_x \psi_\star^{(0)}\right) \\ &+ \partial_T Z + \left(U_\star' Y - \psi_{y\star}^{(0)}\right)\partial_x Z + \left(Y\psi_{xy\star}^{(0)} + \psi_{x\star}^{(1)}\right)\partial_Y Z \\ &+ \partial_t \psi_{xx\star}^{(1)} + Q_\star' \partial_x \psi_\star^{(1)} + \psi_{x\star}^{(0)} \psi_{xxy\star}^{(0)} - \psi_{y\star}^{(0)} q_{x\star}^{(0)} = 0.\end{aligned}$$

A first simplification can be made by noting that the expansion of the linear equation for the Rossby waves around $y_\star$ yields

$$\partial_t q_{y\star}^{(0)} + U_\star' \partial_x q_{y\star}^{(0)} + Q_\star' \partial_x \psi_{y\star}^{(0)} + Q_\star'' \partial_x \psi_\star^{(0)} = 0.$$

Therefore, substituting

$$\Psi_{YY}^{(3)} = Y\psi_{yyy\star}^{(0)} + W(\xi,\eta,\tau,T)$$

and using the transformation (5.10) leads to

$$\partial_\tau W + \partial_T Z + \left[U'_\star\left(\eta - \frac{q^{(0)}_\star}{Q'_\star}\right) - \psi^{(0)}_{y\star}\right]\partial_\xi Z + \left[\eta\left(\psi^{(0)}_{xy\star} + \frac{U'_\star q^{(0)}_{x\star}}{Q'_\star}\right) + \partial_\xi \psi^{(1)}_\star + v_\star\right]\partial_\eta Z + \partial_\tau \psi^{(1)}_{xx\star} + Q'_\star \partial_\xi \psi^{(1)}_\star + \partial(\psi^{(0)}, q^{(0)})_\star = 0, \tag{5.11}$$

where

$$v_\star := -\left(\frac{\psi^{(0)}_{xy\star} q^{(0)}_\star}{Q'_\star} + \frac{\psi^{(0)}_{y\star} q^{(0)}_{x\star}}{Q'_\star} + \frac{U'_\star q^{(0)}_\star q^{(0)}_{x\star}}{{Q'_\star}^2}\right),$$

and $\partial(\psi^{(0)}, q^{(0)})_\star$ denotes the nonlinear advection of the Rossby waves evaluated at $y = y_\star$. Equation (5.11), which governs the fast evolution of W (in the transformed coordinate system), contains secular terms, namely those that are independent of τ. The solvability of (5.11) thus requires that those terms cancel; this provides the slow evolution equation for Z:

$$\partial_T Z + U'_\star \eta\, \partial_\xi Z + \left(\partial_\xi \psi^{(1,s)}_\star + \overline{v}_\star\right)\partial_\eta Z + Q'_\star\, \partial_\xi \psi^{(1,s)}_\star + \overline{\partial(\psi^{(0)}, q^{(0)})_\star} = 0, \tag{5.12}$$

where an overbar denotes the part with zero fast frequency. We can evaluate explicitly

$$\overline{\partial(\psi^{(0)}, q^{(0)})_\star} = -\tfrac{1}{2}\mathrm{Re}\left[(R_1R_2)^* f_+(y_\star)\,\mathrm{e}^{\mathrm{i}k_\star\xi}\right] = -\mathrm{Re}\left(\mathrm{i}k_\star h_\star\,\mathrm{e}^{\mathrm{i}k_\star\xi}\right)$$

and

$$\overline{v}_\star = -\frac{k_\star}{2Q'_\star}\mathrm{Re}\left\{\mathrm{i}(R_1R_2)^*\left[(\psi_1)'\,q_2 + (\psi_2)'\,q_1 + \frac{U'_\star}{Q'_\star}q_1q_2\right]_\star \mathrm{e}^{\mathrm{i}k_\star\xi}\right\} := \mathrm{Re}\left(\overline{v}_{k_\star}\,\mathrm{e}^{\mathrm{i}k_\star\xi}\right).$$

Both terms are independent of T (but in principle dependent on T_2) and oscillatory in ξ with wavenumber $k_\star$.

Equation (5.12) governs the critical-layer dynamics. It is supplemented by a relation between $\psi^{(1,s)}_\star$ and Z provided by the matching condition on the streamwise velocity across the critical layer. From (5.7)–(5.8) and the fact that $\eta \approx Y$ for $Y \to \pm\infty$, it is found that

$$\int_{-\infty}^{+\infty} Z\,\mathrm{d}\eta = \mathrm{Re}\left[\sum_{k=1}^{\infty} C_k\lambda_k(y_\star)\,\mathrm{e}^{\mathrm{i}k\xi}\right], \tag{5.13}$$

where a Cauchy principal value is taken (it can be shown that $Z \sim 1/\eta$ for $\eta \to \pm\infty$). The coefficients C_k are thus determined by

$$C_k = \frac{1}{\lambda_k(y_\star)}\int_{-\infty}^{+\infty} Z_k\,\mathrm{d}\eta, \tag{5.14}$$

where the Fourier coefficients Z_k are defined by

$$Z = \mathrm{Re}\left(\sum_{k=1}^{\infty} Z_k\,\mathrm{e}^{\mathrm{i}k\xi}\right).$$

Finally $\psi^{(1,s)}_\star$ is derived from (5.1) written in the form

$$\psi^{(1,s)}_\star = \mathrm{Re}\left[\phi^{\mathrm{f}}(y_\star) + \mathrm{e}^{\mathrm{i}k_\star\xi}\sum_{k=1}^{\infty} C_k\psi_k(y_\star; y_\star)\,\mathrm{e}^{\mathrm{i}k\xi}\right]. \tag{5.15}$$

Equations (5.12), (5.14) and (5.15) form a closed system which can be integrated forward in time to determine the evolution of Z and of the C_k. This system is similar to that obtained for the forced critical layer by Stewartson (1978) and Warn & Warn (1978), although several differences arise from the transformed cross-stream coordinate, the additional cross-stream advection by $\overline{v}_\star$, and the fact that the forcing is present in the vorticity equation (5.12) but not in the equation for $C_{k_\star}$. The correspondence with the forced critical layer can be seen more clearly by defining a modified streamfunction $D(\xi, T)$ according to

$$\partial_\xi D = \partial_\xi \psi_\star^{(1,\mathrm{s})} + \overline{v}_\star, \tag{5.16}$$

so that (5.12) takes the form

$$\left(\partial_T + U'_\star \eta\, \partial_\xi + \partial_\xi D\, \partial_\eta\right)\left(Z + Q'_\star \eta\right) + F = 0, \tag{5.17}$$

where

$$F(\xi, T) = \overline{\partial(\psi^{(0)}, q^{(0)})_\star} - Q'_\star \overline{v}_\star =: \mathrm{Re}\left[F_{k_\star}(T)\,\mathrm{e}^{\mathrm{i}k_\star \xi}\right]. \tag{5.18}$$

From (5.14), (5.15) and (5.16) we find a relationship between the modified streamfunction D and the critical-layer vorticity Z very similar to that found for the forced critical layer:

$$D_k = \frac{\psi_k(y_\star; y_\star)}{\lambda_k(y_\star)} \int_{-\infty}^{+\infty} Z_k\, \mathrm{d}\eta + \left(\phi_\star^{\mathrm{f}} - \mathrm{i}\frac{\overline{v}_{k_\star}}{k_\star}\right)\delta_{k,k_\star}. \tag{5.19}$$

Only the presence of the internal forcing F in (5.17) prevents a complete analogy between the critical layer generated by Rossby-wave interaction and the forced critical layer.

The ratio $\psi_k(y_\star; y_\star)/\lambda_k(y_\star)$, which is entirely determined by the external parameters of the system (channel geometry and basic flow), governs the strength of the coupling between the critical-layer vorticity and the cross-stream velocity; it is crucial for the nonlinearity within the critical layer. When it tends to infinity, i.e. when $\lambda_k(y_\star) \to 0$, the configuration is that of the Rossby-wave resonance studied by Ritchie (1985) for the forced critical layer. When $\lambda_{k_\star}(y_\star) \to 0$, i.e. when the mode directly forced by the Rossby-wave interaction has continuous streamwise velocity at $y = y_\star$, the above analysis is not valid: one can expect the behaviour of the system to be closer to that of a resonant wave triad in so far as the disturbance excited by the Rossby-wave interaction attains an amplitude comparable to that of the initial waves everywhere in the flow.

The transformed coordinate η defined in (5.10) emerges from our analysis as the natural coordinate for the description of the critical-layer dynamics. It is interesting to remark that it has a straightforward physical interpretation. Since $\delta := -q_\star^{(0)}/Q'$ is the leading-order ($O(\epsilon)$) approximation to the cross-stream displacement induced by the two Rossby waves, $\eta = Y - \delta$ is the cross-stream coordinate in a reference frame advected by the Rossby waves. The critical-layer vorticity is thus advected in the cross-stream direction by the oscillatory velocity field induced by the waves. The extra cross-stream velocity $\overline{v}_\star$ which appears as a consequence of the cross-stream variable transformation is analogous to a Stokes drift.

Although equations (5.12), (5.14) and (5.15), or equivalently (5.17) and (5.19), are sufficient to determine the critical layer dynamics, it is necessary to return briefly to (5.11) in order to derive the coefficients of the equations governing the modification of the Rossby waves in the outer region. When (5.12) is satisfied, the secular terms are removed from (5.11) which can then be solved; this equation shows that W oscillates with frequency $\omega_1 = -\omega_2$ and its harmonics. As a consequence, the dynamics within the critical layer imposes a jump in the y-derivatives of $\psi^{(2,\mathrm{r})}$ and $\psi^{(2,\mathrm{h})}$. This is important not only for $\psi^{(2)}$, but also for $\psi^{(1)}$, since, as already mentioned, the discontinuity in $\psi^{(2,\mathrm{r})}$ appears in the solvability condition determining the evolution of the S_n and hence of $\psi^{(1,\mathrm{r})}$ (see Appendix B for details).

6. A model equation

For the forced critical layer, much insight has been gained by considering the special case often referred to as the SWW solution for which the velocity at the critical level, $\partial_\xi D$, decouples from the critical-layer vorticity Z. The vorticity equation then becomes linear and can be integrated analytically. Stewartson (1978), who derived the analytical solution, initially conceived this situation as an *ad hoc* simplification of the nonlinear critical-layer equation, but Warn & Warn (1978) noted that the decoupling holds exactly (in an asymptotic sense) for certain parameter settings. With our notation, it can indeed be seen from (5.19) that the decoupling occurs provided that

$$\psi_k(y_\star; y_\star) = 0, \quad \forall k.$$

This is possible in the long-wave limit $\mu \to 0$ – because $\psi_k(y, y_\star)$ is then independent of k – provided that β belongs to a discrete set depending on the critical level position $y_\star$. For fixed shear amplitude and dimensional β parameter, this can be achieved by tuning the distance between the channel walls and the critical level.

It is tempting to adapt the SWW solution to the type of critical layer considered in this paper. However, our derivation of the critical-layer evolution equation does not carry over in the long-wave limit upon which the SWW solution relies. This is because we have assumed that the harmonics of the initial Rossby waves are non-resonant, while long waves are non-dispersive and thus have resonant harmonics. In fact, on a time scale $T = O(1)$, long Rossby waves do not maintain their sinusoidal structure in $x - ct$ since they obey a Korteweg–de Vries (KdV) equation (see Redekopp 1977), and the concept of resonant interaction exploited here is not meaningful. (Redekopp & Weidman 1978 analysed the interactions of two long Rossby waves in a shear and found that the evolution of their amplitudes is governed by two coupled KdV equations.) Nevertheless we shall briefly examine the equivalent of the SWW solution for our system, considering it in the spirit of Stewartson (1978), i.e. as a result derived from a model equation rather than a self-consistent solution of the full equations. For the forced critical layer, many features exhibited by the SWW solution are quite generic and relevant to less contrived situations (cf. Haynes 1989). The same is likely to be true for the critical layer studied here. In this case, the SWW solution provides a simple way of assessing the effect of the variable transformation (5.10).

Consider the critical-layer equation (5.17) with

$$D_k = \left(\phi_\star^{\mathrm{f}} - \mathrm{i}\frac{\overline{v}_{k_\star}}{k_\star}\right)\delta_{k,k_\star}$$

in place of (5.19). From this definition and from (5.18) it can be verified that D_k and F_k are in quadrature; we can thus write $D = d\cos(k_\star\xi)$ and $F = f\sin(k_\star\xi)$, where d, f are real constants and $d > 0$ by shifting the origin of ξ. The critical-layer equation then becomes

$$\left[\partial_T + U'_\star\eta\,\partial_\xi - dk_\star\sin(k_\star\xi)\partial_\eta\right]Z - g\sin(k_\star\xi) = 0, \tag{6.1}$$

where $g := k_\star Q'_\star d - f$. Note that we can assume that $g \neq 0$ since $g = 0$ corresponds to the situation mentioned at the end of § 3, where the linear solution remains valid for

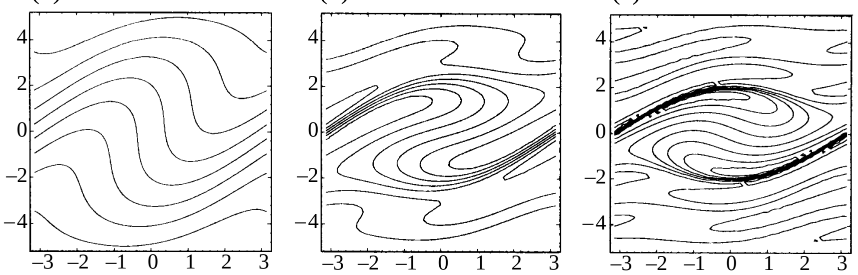

FIGURE 4. Vorticity $Z + a\eta$ with $a = 1/2$ in the (ξ, η)-plane for (*a*) $T = 1$, (*b*) 2.5, (*c*) 4.

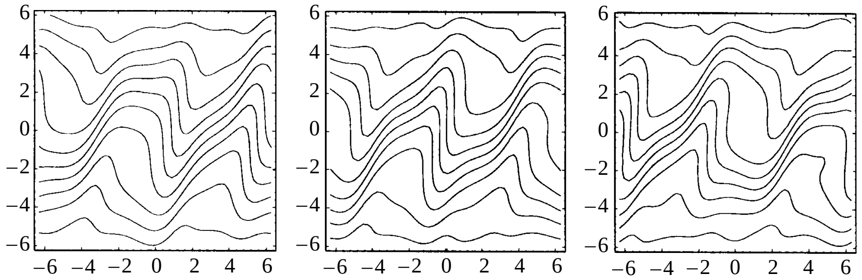

FIGURE 5. Vorticity $Z + aY$ with $a = 1/2$ in the (x, Y)-plane for (*a*) $T = 1$, (*b*) 1.08, (*c*) 1.16. The Rossby-wave period is 0.251 in terms of T.

all time. Now, taking $U'_\star > 0$, and scaling T, ξ, η and Z by $(k_\star^2 U'_\star d)^{-1/2}, k_\star^{-1}, (d/U'_\star)^{1/2}$ and $g(k_\star^2 U'_\star d)^{-1/2}$ leads to the evolution equation in the form

$$(\partial_T + \eta\partial_\xi - \sin\xi\,\partial_\eta)Z - \sin\xi = 0.$$

This is exactly (up to the sign of Z) the equation solved by Stewartson (1978) in terms of elliptic functions. Although the relative vorticity Z is the same as that found in the SWW case, it is not so for the total slow (in the coordinate system (ξ, η)) vorticity $Z + Q'_\star\eta$. In terms of the scaled variables, this total vorticity is indeed given by the expression

$$Z + a\eta, \qquad \text{with} \qquad a := \frac{Q'_\star k_\star d}{g},$$

which is equivalent to that found for the SWW solution only when $a = 1$. The parameter a is fixed by the structure of the interacting Rossby waves. It is equal to 1 if $f = 0$ and thus $F = 0$ in (5.17), in which case (5.17) describes the conservation of the total vorticity.

We illustrate the analytical solution by considering a particular case with $a = 1/2$. Figure 4 shows the time evolution of the scaled vorticity $Z + a\eta$ in the (ξ, η)-plane where it evolves on the slow time scale T only. It displays a cat's eye somewhat different from that obtained for the SWW solution because $a \neq 1$. The evolution in the physical space (x, Y) is computed taking the wavenumbers $k_1 = 0.5$ and $k_2 = -1.5$, so that $k_\star = 1$, consistently with the scaling. The frequencies are $\omega_1 = -\omega_2 = -25\epsilon$ corresponding to a Rossby-wave period of $2\pi/25 = 0.251$ in terms of the slow time T, and the Rossby-wave amplitudes are taken as $R_1 = 0.6$ and $R_2 = 0.8$. All the numerical values have been arbitrarily chosen but they do not affect much the qualitative aspects of the solu-

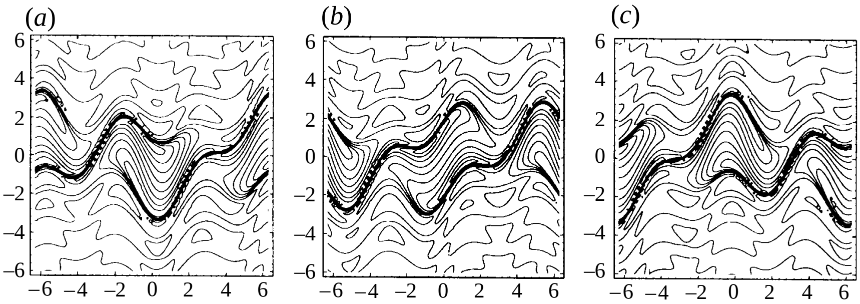

FIGURE 6. Same as figure 5 but for (*a*) $T = 4$, (*b*) 4.08, (*c*) 4.16.

tion. Figures 5 and 6 display the evolution of the scaled vorticity $Z + aY$ in the (x, Y)-plane. (The vorticity directly associated with the Rossby waves should be added to obtain the (scaled) total vorticity in the critical layer which is given by $Z + a\eta$.) T is used as time variable, with $t = \epsilon^{-1}T$, and the vorticity is shown at three different times separated by approximately one-third of the fast period $\omega_1 = -\omega_2$, starting at $T = 1$ (figure 5) and $T = 4$ (figure 6). The figures thus illustrate the fast evolution of the vorticity due to the dependence of Z on t. A complete wavelength of the longest Rossby wave is plotted, so two cat's eyes appear in each panel. The overall picture is that of strongly disturbed cat's eyes, with a propagation of the disturbance in the negative x-direction.

In the framework of the model equation (6.1), the modification of the Rossby-wave amplitudes given by (5.2) and (5.5) can be understood qualitatively by using Stewartson's estimate for

$$\int_{-\infty}^{\infty} Z \, \mathrm{d}\eta = \mathrm{Re}\left[\sum_{k=1}^{\infty} \lambda_k(y_\star) C_k \, \mathrm{e}^{\mathrm{i}k\xi}\right],$$

which he denotes by B, for $T \to \infty$ (equation (4.14) in Stewartson 1978). Integrating this estimate with respect to T from 0 to ∞ yields a finite value. Hence, one can expect the quantity

$$\int_0^T [C_{k_\star}(T')]^* \, \mathrm{d}T', \tag{6.2}$$

which governs the modification of the Rossby waves due to the formation of a critical layer (see (5.5)), to be finite as $T \to \infty$†. This indicates that the Rossby wave modification saturates for large time. Of course this conclusion is obtained under the *ad hoc* assumptions that allowed the derivation of the SWW solution; whether it holds in more general circumstances remains to be verified.

7. Discussion

In this paper we have shown that the interaction between Rossby waves in a shear flow leads to the formation of a nonlinear critical layer, provided that a resonance condition is satisfied. This condition, which ensures that the Rossby waves constitute a resonant triad with a singular mode of the continuous spectrum, is given by an inequality; it is thus easier to satisfy than the standard condition for wave-triad

† From equation (2.30) in Haynes (1989), it can also be noted that the imaginary part of (6.2) is the time-integrated absorptivity introduced by Killworth & McIntyre (1985) who have computed its limiting value as $T \to \infty$.

interaction. A general conclusion can be drawn from our result: in the presence of (regular) Rossby waves, the dynamics of a weakly disturbed, monotonic, unforced shear flow does not necessarily remain linear or weakly nonlinear for all time. This conclusion is similar to that obtained by Brunet & Warn (1990) and Brunet & Haynes (1995) for a non-monotonic shear (parabolic jet with weak vorticity gradient). Both conclusions contrast with Tung's (1983) result which applies to monotonic shear flows without Rossby waves and states that the nonlinearity remains weak if it is so initially.

The Rossby-wave interaction discussed here represents a new mechanism for the development of a nonlinear critical layer, in addition to the mechanisms associated with external forcing (e.g. Stewartson 1978; Warn & Warn 1978), marginal instability (e.g. Brown & Stewartson 1978; Hickernell 1984; see also Goldstein 1994 and references therein), or with non-monotonic shear (Brunet & Warn 1990; Brunet & Haynes 1995). Recently, Balmforth, del-Castillo-Negrete & Young (1997) derived a critical-layer equation for a monotonic flow containing a vorticity defect, i.e. a narrow region where the vorticity gradient is non-zero. In that situation, the critical-layer width is fixed by that of the vorticity defect; and if there is no forcing nor instability the amplitude of the initial disturbance can be chosen small enough so that the evolution remains linear for all time. For the Rossby-wave interaction or the non-monotonic shear, however, the critical-layer width is determined by the amplitude of the initial disturbance and the strongly nonlinear behaviour is inevitable.

Two different approaches are employed to examine the nonlinear evolution of the interacting Rossby waves and of the continuous spectrum: a weakly nonlinear analysis and a critical-layer analysis. It is instructive to compare some aspects of these two approaches. In deriving the weakly nonlinear amplitude equations, we mentioned that the system obtained by direct use of the normal-mode expansion cannot be easily truncated. This is because the linear evolution of a superposition of singular modes leads to a linearly growing gradient of vorticity, so the nonlinear terms corresponding to the interaction of Rossby waves with singular modes are themselves linearly growing in time. We introduced the transformation (4.2) of the dependent variable q in order to remove these terms from the evolution equation, but pointed out that the transformation is subject to condition (4.3) which is likely to be violated for long time.

In the matched asymptotics formalism used in the critical-layer analysis, the terms corresponding to the interaction between Rossby waves and singular modes play a particular role too. In that formalism, the vorticity associated with singular modes is given at leading order by $Z = \Psi^{(2)}_{YY} - \psi^{(0)}_{yy\star}$, and the interaction with Rossby waves appears as the term $\partial_x \psi^{(0)}_\star \partial_Y \Psi^{(2)}_{YY}$ in (5.9). This term is eliminated using the transformation of the independent variables (5.10), which we interpreted as the use of a reference frame advected by the Rossby waves. In this reference frame, the advection of the singular-mode vorticity by the waves disappears and the corresponding secularity is removed. (Note that the same variable transformation was previously introduced by G. Brunet 1989, unpublished notes, in a study of the interaction between a sheared disturbance and a Rossby wave.)

The two variable transformations we have introduced, that of the dependent variable (4.2) and that of the independent variables (5.10), are in fact closely connected: for $Q' = Q'_\star$ (as is approximately the case within the critical layer), it can be verified that (4.2) is equivalent to

$$q(x, y, t) = \tilde{q}(x, y + \epsilon q/Q'_\star, t)$$

to first order in ϵ, assuming (4.3). (This transformation is the same as the one introduced by Zakharov & Piterbarg (1988) to derive canonical Hamiltonian equations for Rossby waves.) When q is approximated by its leading-order component $q^{(0)}$, the transformation becomes equivalent to (5.10) since $y + \epsilon q^{(0)}/Q'_\star = y_\star + \epsilon\eta$.

The above discussion suggests the possibility of extending (4.2) to remove the constraint (4.3) which limits the validity of our weakly nonlinear expansion. Using the orthogonality relations (A 5), the disturbance vorticity can always be unambiguously decomposed into a Rossby wave part, $q^{(\mathrm{r})}$ say, and a part associated with the singular modes. Now, when the nonlinearity is weak, the independent variable transformation $\tilde{y} := y + \epsilon q^{(\mathrm{r})}/Q'$ is in general well defined, because the y-derivatives of $q^{(\mathrm{r})}$ remains bounded in the linear approximation. Deriving an evolution equation for $\tilde{q}(x, \tilde{y}, t) := q(x, y, t)$ with $\tilde{y}$ as independent cross-stream coordinate, one finds a weakly nonlinear equation similar to (4.4): it has the same linear part and the advection of the singular modes by the Rossby waves is absent (but the self-advection of the singular modes remains). The normal-mode expansion can then be used for $\tilde{q}$ and again leads to amplitude equations without linearly growing terms.

The critical-layer analysis developed in this paper is very similar to that of Stewartson (1978) and Warn & Warn (1978), except for additional elements related to the presence of Rossby waves at leading order. Among these, the modification of the Rossby-wave amplitudes induced by their interaction is particularly interesting. It is a reflection of the fact that we are considering an unforced problem: the Rossby waves are acting as forcing for the critical layer which, in turn, affects their dynamics. The derivation of the equations governing this modification is fairly tedious, but, apart from its technical details, it is important since it leads to the conclusion that the modification of the waves represents only a small $O(\epsilon)$ correction relative to the initial Rossby-wave amplitudes. Such a small modification is not incompatible with the fact that the Rossby waves provide most of the energy for the critical-layer formation because of the small width of the critical layer.

The analogy between the problem considered here and the forced critical layer along with the use of the SWW solution provide a qualitative understanding of the behaviour of the critical layer generated by Rossby-wave interaction. Within the critical layer, we can expect a complex wrapping up of the vorticity by the streamline pattern generated by the basic flow, the two Rossby waves, and the cross-stream velocity $\partial_\xi \psi^{(1,\mathrm{s})}$, all of these components having the same importance. The slow evolution of the flow is dominated by cat's eye structures, but fast oscillations, associated directly with the presence of the Rossby waves, and indirectly with the advection of the critical-layer vorticity, are superposed on the slow motion. Outside the critical layer, the motion remains essentially linear, although at $O(\epsilon)$ all harmonics are excited because of the velocity jump imposed by the evolution inside the critical-layer. Of course, only numerical simulations, of the critical layer equation or of the original nonlinear equation, can give a detailed picture of the critical layer dynamics generated by Rossby-wave interaction. We leave such simulations for future work.

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Appendix A. Normal modes

Introducing normal-mode solutions of the form

$$\psi(x,y,t) = \mathrm{Re}\left[\psi_k(y,t)\,\mathrm{e}^{\mathrm{i}kx}\right] = \mathrm{Re}\left[\psi_k(y)\,\mathrm{e}^{\mathrm{i}k(x-ct)}\right]$$

into the linearization of (2.1) leads to the Rayleigh–Kuo equation which, together with the boundary conditions $\psi_k(0) = \psi_k(1) = 0$, defines an eigenvalue problem, with the phase velocity c as eigenvalue. With the assumption $Q' > 0$ the solutions are stable modes of two types: (regular) Rossby waves, with $c < U_{\mathrm{m}}$, and singular modes, with $U_{\mathrm{m}} < c < U_{\mathrm{M}}$ (e.g. Drazin, Beaumont & Coaker 1982).

Rossby waves are solutions of the homogeneous Rayleigh–Kuo equation; for each k, they form a discrete spectrum with eigenvalues $c = c_{k,n}$ and eigenfunctions $\psi_{k,n}(y)$, $n = 1, 2, \ldots$.

Singular modes exist for each $U_{\mathrm{m}} < c < U_{\mathrm{M}}$, or equivalently for each critical-level position y_c defined by $U(y_c) = c$. They are solutions of the singular equation

$$\frac{\partial^2}{\partial y^2}\psi_k(y;y_c) - k^2\psi_k(y;y_c) + \frac{Q'(y)}{U(y)-U(y_c)}\psi_k(y;y_c) = \lambda_k(y_c)\delta(y-y_c), \qquad \text{(A 1)}$$

with homogeneous boundary conditions. Here, $\lambda_k(y_c)$ is a velocity jump, i.e.

$$\lambda_k(y_c) = \left[\frac{\partial\psi_k(y;y_c)}{\partial y}\right]_{y_c^-}^{y_c^+}, \qquad \text{(A 2)}$$

and it is determined by the normalization chosen for $\psi_k(y;y_c)$. The streamfunction $\psi_k(y;y_c)$ is continuous at $y = y_c$, but the corresponding vorticity must be interpreted as the distribution

$$q_k(y;y_c) = \lambda_k(y_c)\delta(y-y_c) - \mathscr{P}\frac{Q'(y)}{U(y)-U(y_c)}\psi_k(y;y_c),$$

where $\mathscr{P}$ denotes the Cauchy principal value. The structure $\psi_k(y;y_c)$ can be found by solving a regular integral equation (Kamp 1991; Balmforth & Morrison 1998), or by using a combination of two independent solutions of the homogeneous version of (A 1) on each side of the critical level. Near the critical level $y_\star$, the singular mode streamfunction is given by

$$\psi_k(y;y_c) = \psi_k(y_c;y_c)\left[1 - \frac{Q'(y_c)}{U'(y_c)}(y-y_c)\ln|y-y_c|\right]$$
$$+ v_k(y_c)(y-y_c) + \tfrac{1}{2}\lambda_k(y_c)|y-y_c| + O[(y-y_c)^2\ln|y-y_c|], \qquad \text{(A 3)}$$

where $v_k(y_c)$ is a smooth function determined by the boundary conditions.

Together with the Rossby modes, the singular modes can be used to expand any Fourier component k of the streamfunction and vorticity according to

$$\left.\begin{aligned} \psi_k(y,t) &= \sum_n A_{k,n}(t)\psi_{k,n}(y)\,\mathrm{e}^{-\mathrm{i}\omega_{k,n}t} + \int_0^1 A_k(y_c;t)\psi_k(y;y_c)\,\mathrm{e}^{-\mathrm{i}kU(y_c)t}\,\mathrm{d}y_c, \\ q_k(y,t) &= \sum_n A_{k,n}(t)q_{k,n}(y)\,\mathrm{e}^{-\mathrm{i}\omega_{k,n}t} + \int_0^1 A_k(y_c;t)q_k(y;y_c)\,\mathrm{e}^{-\mathrm{i}kU(y_c)t}\,\mathrm{d}y_c. \end{aligned}\right\} \tag{A 4}$$

The amplitudes $A_{k,n}(t)$ and $A_k(y_c;t)$ may be deduced from $\psi_k(y,t)$ or $q_k(y,t)$ using the orthogonality relations derived by Balmforth & Morrison (1998), which may be written

$$\int_0^1 \bar{q}_{k,n}(y)\,q_{k,n'}(y)\,\mathrm{d}y = \delta_{n,n'} \quad \text{and} \quad \int_0^1 \bar{q}_k(y;y_c)\,q_k(y;y_c')\,\mathrm{d}y = \delta(y_c - y_c'), \tag{A 5}$$

with well-defined functions $\bar{q}_{k,n}(y)$ and $\bar{q}_k(y;y_c)$ (see Balmforth & Morrison 1998; see also Vanneste 1996). By construction (A 4) with constant amplitudes $A_{k,n}(t)$ and $A_k(y_c;t)$ provides an exact solution to the linearized equations of motion. Note that for constant $A_k(y_c;t)$ the streamfunction associated with singular modes decays like t^{-2} for large t, for $\psi_k(y;y_c)$ is continuously differentiable only once.

Appendix B. Modification of the Rossby waves

Evolution equations for $S_1(T)$ and $S_2(T)$ determining the small perturbation of the Rossby-wave amplitudes (see (5.2)) are found from the solvability conditions for $\psi^{(2,\mathrm{r})}$. From (5.3), one can see that the forcing for $\psi^{(2,\mathrm{r})}$ is given by

$$-\partial_T q^{(1,\mathrm{r})} - \partial(\psi^{(1,\mathrm{s})}, q^{(0)}) - \partial(\psi^{(0)}, q^{(1,\mathrm{s})})$$

plus the part of $-\partial(\psi^{(0)}, q^{(1,\mathrm{h})}) - \partial(\psi^{(1,\mathrm{h})}, q^{(0)})$ oscillating with frequencies ω_1 and ω_2. This forcing involves all wavenumbers, but only the part with wavenumbers k_1 and k_2 is of interest here since it is associated with resonance. Let $\Phi_1(y,T)$ and $\Phi_2(y,T)$ be the components of $\psi^{(2,\mathrm{r})}$ with wavenumber k_1 and k_2, respectively. They satisfy

$$\mathrm{i}k_n\left[(U-c_n)\left(\partial_{yy}^2\Phi_n - \mu^2(k_n)^2\Phi_n\right) + Q'\Phi_n\right] = -\partial_T S_n\,q_n - v_n, \quad n = 1,2, \tag{B 1}$$

where $v_n(y,T)$ contains the nonlinear (Jacobian) terms. In particular, v_n contains contributions from $\partial(\psi^{(1,\mathrm{s})}, q^{(0)}) + \partial(\psi^{(0)}, q^{(1,\mathrm{s})})$ which are singular, so that $v_n \sim (y - y_\star)^2$ near $y_\star$.

To find the solvability condition, we employ a method introduced by Benney & Maslowe (1975). Let Φ_n^a and Φ_n^b be two independent homogeneous solutions of (B 1). For convenience, we take them such that $\Phi_n^a(y_\star) = 0$. The general solution of (B 1) is then written

$$\Phi_n = \begin{cases} a_n^-\,\Phi_n^a(y) + b_n^-\,\Phi_n^b(y) - \partial_T S_n\,u_n(y) - w_n(y,T), & y < y_\star \\ a_n^+\,\Phi_n^a(y) + b_n^+\,\Phi_n^b(y) - \partial_T S_n\,u_n(y) - w_n(y,T), & y > y_\star, \end{cases} \tag{B 2}$$

where u_n and w_n are non-homogeneous solutions corresponding to q_n and v_n, respectively. They can be determined for instance by using the method of variation of parameters. Near the critical level, $w_n \sim \ln|y - y_\star|$. The piecewise definition (B 2) of the Φ_n is essential because the dynamics inside the critical layer imposes a jump in $\partial_y\psi^{(2,\mathrm{r})}$ at the critical level. Let $j_n(T)$, $n = 1,2$, be the parts of this jump with space–time dependence $\exp[\mathrm{i}(k_n x - \omega_n t)]$, i.e.

$$j_n(T) = \left[\frac{\partial \Phi_n}{\partial y}\right]_{y_\star^-}^{y_\star^+}.$$

The j_n are determined by the dynamics inside the critical layer and will be evaluated explicitly below. Taking the jumps j_n into account and enforcing continuity at $y = y_\star$, we can rewrite (B 2) as

$$\Phi_n = a_n \Phi_n^a(y) + b_n \Phi_n^b(y) \pm \frac{j_n(T)}{2(\Phi_n^a)'_\star} \Phi_n^a(y) - \partial_T S_n\, u_n(y) - w_n(y, T), \tag{B 3}$$

where $(\Phi_n^a)'_\star = \mathrm{d}\Phi_n^a/\mathrm{d}y|_{y_\star}$ and where the $+$ $(-)$ sign corresponds to $y > y_\star$ $(y < y_\star)$. The constants a_n and b_n should be determined by applying the homogeneous boundary conditions at $y = 0, 1$. However, the determinant $\Phi_n^a(1)\Phi_n^b(0) - \Phi_n^a(0)\Phi_n^b(1) = 0$ because (B 1) has a non-trivial homogeneous solution (the Rossby wave with frequency ω_n). Therefore, a solution exists only if a compatibility condition of the form

$$\partial_T S_n + P_n(T) = 0, \quad n = 1, 2, \tag{B 4}$$

is satisfied, where

$$P_n(T) = \frac{1}{\Phi_n^a(1) u_n(0) - \Phi_n^a(0) u_n(1)} \left[\Phi_n^a(1) w_n(0, T) - \Phi_n^a(0) w_n(1, T) + \frac{j_n(T)}{(\Phi_n^a)'_\star} \Phi_n^a(1) \Phi_n^a(0)\right]. \tag{B 5}$$

The two equations for (B 4) govern the evolution of the modification $\psi^{(1,\mathrm{r})}$ of the Rossby waves.

We now turn to the derivation of the velocity jumps j_1 and j_2. These jumps are associated with the $O(\epsilon)$ critical-layer vorticity W which evolves according to (5.11). Precisely, writing

$$W = \psi_{yy\star}^{(1,\mathrm{r})} + \mathrm{Re}\left[W_1(\eta, T)\, \mathrm{e}^{\mathrm{i}(k_1 \xi - \omega_1 \tau)} + W_2(\eta, T)\, \mathrm{e}^{\mathrm{i}(k_2 \xi - \omega_2 \tau)}\right] + \cdots, \tag{B 6}$$

where $\cdots$ denotes terms with wavenumbers and frequencies different from (k_1, ω_1) and (k_2, ω_2), leads to

$$j_n = \int_{-\infty}^{+\infty} W_n \, \mathrm{d}\eta. \tag{B 7}$$

Explicit expressions for W_1 and W_2 are readily derived from (5.11); they are given by

$$\begin{aligned} W_n = \frac{1}{2\omega_n} \Bigg\{ & k_\star \left[\frac{U'}{Q'} q_m + (\psi_m)'\right]_\star \left(R_m Z_{k_\star}\right)^* \\ & - k_m \left(\eta \left[\frac{U'}{Q'} q_m + (\psi_m)'\right]_\star (R_m)^* + \psi_m (S_m)^*\right) \left(\partial_\eta Z_{k_\star}\right)^* \Bigg\}, \end{aligned} \tag{B 8}$$

where $m = 3 - n$. Introducing this result into (B 7), integrating by parts and using (2.2) and (5.14) yields

$$j_n = -\frac{1}{2\omega_n} k_n \lambda_{k_\star}(y_\star) \left[\frac{U'}{Q'} q_m + (\psi_m)'\right]_\star (R_m C_{k_\star})^*. \tag{B 9}$$

This expression (for $n = 1, 2$) completes our determination of the coefficients P_n given by (B 5), which govern the modification of the Rossby waves. From (B 5) and (B 9) one sees that the P_n have the generic form

$$P_n = c_n + d_n[C_{k_\star}(T)]^*,$$

where c_n and d_n are complex constants. The dependence on the critical-layer-controlled quantity $C_{k_\star}$ stems from two effects: the dependence of the outer streamfunction $\psi^{(1,\mathrm{r})}$ on $C_{k_\star}$, and the direct role of the inner dynamics in the solvability condition as embodied in the velocity jumps j_n.

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